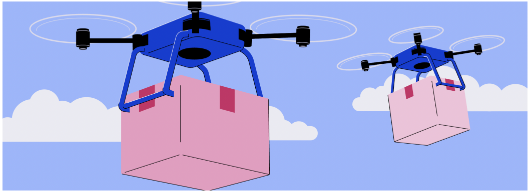
Safe Control of Partially-Observed Linear Time-Varying Systems with Minimal Worst-Case Dynamic Regret

Hongyu Zhou and Vasileios Tzoumas

MICHIGAN ENGINEERING
UNIVERSITY OF MICHIGAN

Drone Delivery

Goal: Deliver a package in a moving vehicle.



Complication: Partially-observed systems; Unpredictable wind disturbances.

Problem: How can the drone choose collision-free optimal control actions?[1]

[1] Ackerman, IEEE Spectrum' 13

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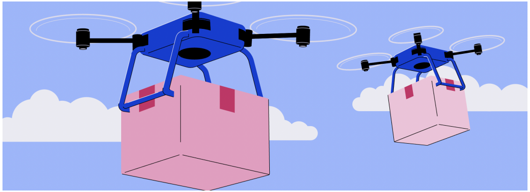

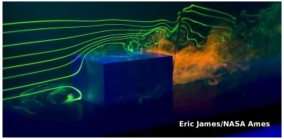

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Safe Optimal Control Problems

Target Tracking

Goal: Minimize distance to a target that moves in a cluttered environment.

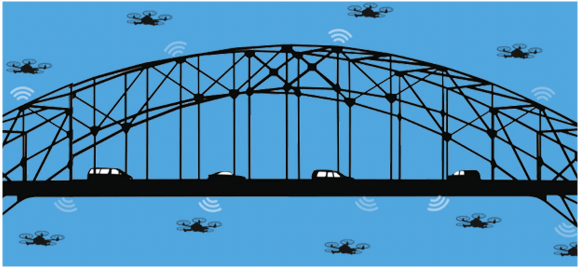
Complication: Partially-observed systems; Unpredictable wind disturbances.

Problem: How can the drone choose collision-free optimal control actions?[1]

[1] Chen, Liu, Shen, IROS' 16

Inspection and Maintenance

Goal: Inspect and repair facilities using onboard cameras.



Complication: Partially-observed systems; Unpredictable wind disturbances.

Problem: How can the drone choose collision-free optimal control actions?[1]

[1] Seneviratne, Dammika, et al. Acta Imeko '18

All above scenarios are optimal control problems with safety constraints

Goal: Find control input to minimize loss subject to system dynamics and safety constraints:

$$\min_{\text{control input}} \quad \text{loss} \qquad \textit{e.g.}\text{, tracking error}$$
$$\text{subject to} \quad \text{system dynamics;}$$
$$\text{safety constraints.}$$

Safe Control of Partially-Observed Linear Time-Varying Systems

All above scenarios are optimal control problems with safety constraints

Goal: Find control input to minimize loss subject to system dynamics and safety constraints:

$$\min_{u_t} \quad \sum_{t=0}^{T-1} x_{t+1} Q x_{t+1}^\top + u_t R u_t^\top$$

$$\text{subject to} \quad x_{t+1} = A_t x_t + B_t u_t + w_t,$$

$$y_t = C_t x_t + e_t;$$

$$x_t \in \mathcal{S}_t, \; u_t \in \mathcal{U}_t.$$

known system matrices

constraints

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\end{aligned}$$

known system matrices

polytopic constraints
for simplicity

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$$\begin{aligned} \min_{\mathbf{u}} \quad & \mathbf{x}^\top \mathcal{Q}\mathbf{x} + \mathbf{u}^\top \mathcal{R}\mathbf{u} \\ \text{subject to} \quad & \mathbf{x} = \mathcal{Z}\mathcal{A}\mathbf{x} + \mathcal{Z}\mathcal{B}\mathbf{u} + \mathbf{w}, \\ & \mathbf{y} = \mathcal{C}\mathbf{x} + \mathbf{e}; \\ & \mathbf{H}\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \leq \mathbf{h}, \end{aligned}$$

where:

$$\mathbf{x} \triangleq \begin{bmatrix} x_0 \\ x_1 \\ \dots \\ x_{T-1} \end{bmatrix}, \mathbf{u} \triangleq \begin{bmatrix} u_0 \\ x_1 \\ \dots \\ u_{T-1} \end{bmatrix}, \mathbf{w} \triangleq \begin{bmatrix} x_0 \\ w_1 \\ \dots \\ w_{T-2} \end{bmatrix}, \mathbf{y} \triangleq \begin{bmatrix} y_0 \\ y_1 \\ \dots \\ y_{T-1} \end{bmatrix}, \mathbf{e} \triangleq \begin{bmatrix} e_0 \\ e_1 \\ \dots \\ e_{T-1} \end{bmatrix}, \dots$$

All above scenarios are optimal control problems with safety constraints

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$$
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& \mathbf{H} \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \leq \mathbf{h},
\end{aligned}
$$

and where: $\mathcal{A} \triangleq \text{blkdiag}\left(A_0, A_1, \ldots, A_{T-2}, \mathbf{0}\right)$, $\mathcal{B} \triangleq \text{blkdiag}\left(B_0, B_1, \ldots, B_{T-2}, \mathbf{0}\right)$,

$$
\mathcal{C} \triangleq \text{blkdiag}\left(C_0, C_1, \ldots, C_{T-1}\right),\ \mathcal{Z} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{I} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \ldots & \mathbf{I} & \mathbf{0} \end{bmatrix}.
$$

All above scenarios are optimal control problems with safety constraints

Goal: Find control input to minimize loss subject to system dynamics and safety constraints:

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Difficulty:

- The system is partially-observed;
- The noise $\mathbf{w}$ and $\mathbf{e}$ can be unknown and unstructured.

Suboptimality Metric against Optimal Control Policies in Hindsight

Definition (Dynamic Regret)

Assume a lookahead time horizon of operation T, and loss functions c_t, $t = 1, \ldots, T$. Then, dynamic regret is defined as

$$\text{Regret}_T(\mathbf{w}, \mathbf{e}, \mathbf{u}) = \left(\mathbf{x}^\top \mathcal{Q}\mathbf{x} + \mathbf{u}^\top \mathcal{R}\mathbf{u}\right) - \left({\mathbf{x}^*}^\top \mathcal{Q}\mathbf{x}^* + {\mathbf{u}^*}^\top \mathcal{R}\mathbf{u}^*\right),$$

where $\mathbf{x}^*$ and $\mathbf{u}^*$ are the optimal trajectory and control input in hindsight given the noise realization due to $\mathbf{u}$.

Remark:

- The dynamic regret is sublinear if $\lim_{T\to\infty} \frac{\text{Regret}_T}{T} \to 0$, which implies $c_t\left(x_{t+1}, u_t\right) - c_t\left(x^*_{t+1}, u^*_t\right) \to 0$ as $T \to \infty$.

Suboptimality Metric against Optimal Control Policies in Hindsight

Definition (Worst-Case Dynamic Regret)

The worst-case-regret is defined as

$$\text{Worst-Case-Regret}_T(\mathbf{u}) \triangleq \max_{\|\mathbf{w}\|_2^2 + \|\mathbf{e}\|_2^2 \leq r^2} \text{Regret}_T(\mathbf{w}, \mathbf{e}, \mathbf{u}),$$

where r is a given positive number.

Remark:

- The worst-case dynamic regret provides a robust performance guarantee by assuming the worst-case noise.[2]

[2] Goel et al., '20; Sabag et al., ACC '21; Goel et al., L4DC '21; Martin et al., L4DC '22; Didier et al., L-CSS '22; ...

Safe Control of Partially-Observed Linear Time-Varying Systems

Problem

Find control $\mathbf{u}$ *for the partially-observed LTV system that minimizes worst-case dynamic regret subject to safety constraints, i.e.,*

$$
\begin{aligned}
\min_{\mathbf{u}} \quad & \text{Worst-Case-Regret}_T(\mathbf{u}) \\
\textit{subject to} \quad & \mathbf{x} = \mathcal{Z}\mathcal{A}\mathbf{x} + \mathcal{Z}\mathcal{B}\mathbf{u} + \mathbf{w}, \\
& \mathbf{y} = \mathcal{C}\mathbf{x} + \mathbf{e}; \\
& \mathbf{H}\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \leq \mathbf{h}.
\end{aligned}
\tag{1}
$$

Remark: The problem generalizes to the partially-observable case the optimal control problem in Goel et al., '20; Sabag et al., ACC '21; Martin et al., L4DC '22; Didier et al., L-CSS '22.

Output-Feedback control policy:

$$u_t = \sum_{k=0}^{t} K_{t,k} y_k, \quad t \in \{0, \ldots, T-1\},$$

where $K_{t,k}$ are control gains to be designed.

Compact form:

$$\mathbf{u} = \mathcal{K}\mathbf{y},$$

where

$$\mathcal{K} \triangleq \begin{bmatrix} K_{0,0} & \mathbf{0} & \ldots & \mathbf{0} \\ K_{1,0} & K_{1,1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \mathbf{0} \\ K_{T-1,0} & K_{T-1,1} & \ldots & K_{T-1,T-1} \end{bmatrix}.$$

Assumptions

Assumption (Bounded Noise)

$\mathbf{w} \in \mathbb{W} \triangleq \{\mathbf{w} \mid \mathbf{H}_w \mathbf{w} \leq \mathbf{h}_w\}$ *and* $\mathbf{e} \in \mathbb{E} \triangleq \{\mathbf{e} \mid \mathbf{H}_e \mathbf{e} \leq \mathbf{h}_e\}$ *with* $\mathbf{H}_w$, $\mathbf{H}_e$, $\mathbf{h}_\mathbf{w}$, *and* $\mathbf{h}_\mathbf{e}$ *given.*

Remark:

- We assume no stochastic model for the noise $\mathbf{w}$ and $\mathbf{e}$: the noise may even be adversarial.

Example:

- Wind disturbances of bounded magnitude, whose evolution may not be governed by a known stochastic model.

Regret Optimal Control[3]

- selects control inputs over a lookahead horizon;
- assumes worst-case noise.

[3] Goel et al., '20; Sabag et al., ACC '21; Goel et al., L4DC '21; Martin et al., L4DC '22; Didier et al., L-CSS '22; ...

Closest Related Work

Regret Optimal Control[3]

- selects control inputs over a lookahead horizon;
- assumes worst-case noise.

Online Learning for Control[4]

- selects control inputs based on past information only;
- consider non-stochastic noise.

[3] Goel et al., '20; Sabag et al., ACC '21; Goel et al., L4DC '21; Martin et al., L4DC '22; Didier et al., L-CSS '22; ...

[4] Agarwal et al., ICML '19; Hazan et al., ALT '20; Li et al., AAAI '21; Zhao et al., AISTATS '22; Gradu et al., L4DC '23; Zhou et al., CDC '23; ...

Closest Related Work

Regret Optimal Control[3]

- selects control inputs over a lookahead horizon;
- assumes worst-case noise.

Online Learning for Control[4]

- selects control inputs based on past information only;
- consider non-stochastic noise.

BUT:

- considers no safety constraints or
- considers fully-observed systems.

[3] Goel et al., '20; Sabag et al., ACC '21; Goel et al., L4DC '21; Martin et al., L4DC '22; Didier et al., L-CSS '22; ...

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Preliminary: System Level Synthesis for Partially-Observable LTV Systems

Proposition

There exists a lower triangular block-matrix $\mathcal{K} = \mathbf{\Phi}_{ue} - \mathbf{\Phi}_{uw}\mathbf{\Phi}_{xw}^{-1}\mathbf{\Phi}_{xe}$ *such that*

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_{xw} & \mathbf{\Phi}_{xe} \\ \mathbf{\Phi}_{uw} & \mathbf{\Phi}_{ue} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{e} \end{bmatrix},$$

holds true if and only if $\mathbf{\Phi}_{xw}$, $\mathbf{\Phi}_{xe}$, $\mathbf{\Phi}_{uw}$, *and* $\mathbf{\Phi}_{ue}$ *are:*

- *lower triangular block-matrices; and*
- *lie in the affine subspace*

$$\begin{bmatrix} \mathbf{I} - \mathcal{Z}\mathcal{A} & -\mathcal{Z}\mathcal{B} \end{bmatrix} \mathbf{\Phi} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix},$$

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where $\mathbf{\Phi} \triangleq \begin{bmatrix} \mathbf{\Phi}_{xw} & \mathbf{\Phi}_{xe} \\ \mathbf{\Phi}_{uw} & \mathbf{\Phi}_{ue} \end{bmatrix}$ *is the response matrix.*

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} convex constraints on $\mathbf{\Phi}$

where $\mathbf{\Phi} \triangleq \begin{bmatrix} \mathbf{\Phi}_{xw} & \mathbf{\Phi}_{xe} \\ \mathbf{\Phi}_{uw} & \mathbf{\Phi}_{ue} \end{bmatrix}$ *is the response matrix.*

Semi-Definite Program Reformulation

Theorem

The problem in eq. (1) is equivalent to the Semi-Definite Program

$$
\begin{aligned}
&\min_{\boldsymbol{\Phi}, \boldsymbol{Z}, \lambda} \quad \lambda \quad \textit{subject to:} \\
&\boldsymbol{\Phi}_{xw}, \boldsymbol{\Phi}_{xe}, \boldsymbol{\Phi}_{uw}, \boldsymbol{\Phi}_{ue} \ \textit{being lower block triangular;} \\
&\begin{bmatrix} \mathbf{I} - \mathcal{Z}\mathcal{A} - \mathcal{Z}\mathcal{B} \end{bmatrix} \boldsymbol{\Phi} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \boldsymbol{\Phi} \begin{bmatrix} \mathbf{I} - \mathcal{Z}\mathcal{A} \\ -\mathcal{C} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}, \\
&\mathbf{Z}^\top \begin{bmatrix} \mathbf{h}_w \\ \mathbf{h}_e \end{bmatrix} \leq \mathbf{h}, \ \mathbf{H}\boldsymbol{\Phi} = \mathbf{Z}^\top \begin{bmatrix} \mathbf{H}_w & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_e \end{bmatrix}, \ \mathbf{Z}_{ij} \geq 0; \\
&\lambda > 0, \ \begin{bmatrix} \mathbf{I} & \mathcal{D}^{\frac{1}{2}}\boldsymbol{\Phi} \\ \boldsymbol{\Phi}^\top \mathcal{D}^{\frac{1}{2}} & \lambda \mathbf{I} + (\boldsymbol{\Phi}^c)^\top \mathcal{D} \boldsymbol{\Phi}^c \end{bmatrix} \succeq 0,
\end{aligned}
\tag{2}
$$

where $\mathcal{D} \triangleq \mathrm{blkdiag}(\mathcal{Q}, \mathcal{R})$, $\mathbf{Z}$ *are the dual variables,* $\boldsymbol{\Phi}^c$ *is the response corresponding to the optimal clairvoyant controller.*

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$$\min_{\boldsymbol{\Phi}, \boldsymbol{Z}, \lambda} \quad \lambda \quad \textit{subject to:}$$

change variables from $\mathcal{K}$ to $\boldsymbol{\Phi}$

$$\begin{cases} \boldsymbol{\Phi}_{xw}, \boldsymbol{\Phi}_{xe}, \boldsymbol{\Phi}_{uw}, \boldsymbol{\Phi}_{ue} \text{ being lower block triangular;} \\ \begin{bmatrix} \mathbf{I} - \mathcal{Z}\mathcal{A} - \mathcal{Z}\mathcal{B} \end{bmatrix} \boldsymbol{\Phi} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \boldsymbol{\Phi} \begin{bmatrix} \mathbf{I} - \mathcal{Z}\mathcal{A} \\ -\mathcal{C} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}, \end{cases}$$

$$\mathbf{Z}^\top \begin{bmatrix} \mathbf{h}_w \\ \mathbf{h}_e \end{bmatrix} \leq \mathbf{h}, \ \mathbf{H}\boldsymbol{\Phi} = \mathbf{Z}^\top \begin{bmatrix} \mathbf{H}_w & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_e \end{bmatrix}, \ \mathbf{Z}_{ij} \geq 0; \tag{2}$$

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use duality for safety constraints

$$\begin{cases} \mathbf{Z}^\top \begin{bmatrix} \mathbf{h}_w \\ \mathbf{h}_e \end{bmatrix} \leq \mathbf{h}, \ \mathbf{H}\boldsymbol{\Phi} = \mathbf{Z}^\top \begin{bmatrix} \mathbf{H}_w & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_e \end{bmatrix}, \ \mathbf{Z}_{ij} \geq 0; \end{cases} \tag{2}$$

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Schur complement

$$\lambda > 0, \ \begin{bmatrix} \mathbf{I} & \mathcal{D}^{\frac{1}{2}} \mathbf{\Phi} \\ \mathbf{\Phi}^\top \mathcal{D}^{\frac{1}{2}} & \lambda \mathbf{I} + (\mathbf{\Phi}^c)^\top \mathcal{D} \mathbf{\Phi}^c \end{bmatrix} \succeq 0,$$

Generalization of steps in Goe et al., L4DC '21 and Martin et al., L4DC '22

where $\mathcal{D} \triangleq \mathrm{blkdiag}(\mathcal{Q}, \mathcal{R})$, $\mathbf{Z}$ *are the dual variables,* $\mathbf{\Phi}^c$ *is the response corresponding to the optimal clairvoyant controller.*

Algorithm

Initialization: Time horizon T; system matrices $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$; cost matrices $\mathcal{Q}$ and $\mathcal{R}$; noise's domain sets $\mathbb{W}$ and $\mathbb{E}$; upper bound r to the noise' total magnitude.

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Output: Output-feedback control gains $\mathcal{K}$.

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1. $\{\mathbf{\Phi}_{xw}, \mathbf{\Phi}_{xe}, \mathbf{\Phi}_{uw}, \mathbf{\Phi}_{ue}\} \leftarrow$ Solve the Semi-Definite Program in eq. (2);

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1. $\{\mathbf{\Phi}_{xw}, \mathbf{\Phi}_{xe}, \mathbf{\Phi}_{uw}, \mathbf{\Phi}_{ue}\} \leftarrow$ Solve the Semi-Definite Program in eq. (2);
2. $\mathcal{K} \leftarrow \mathbf{\Phi}_{ue} - \mathbf{\Phi}_{uw}\mathbf{\Phi}_{xw}^{-1}\mathbf{\Phi}_{xe}$.

Numerical Evaluation on Synthetic Partially-Observed LTV Systems

Setup:

- linear system:

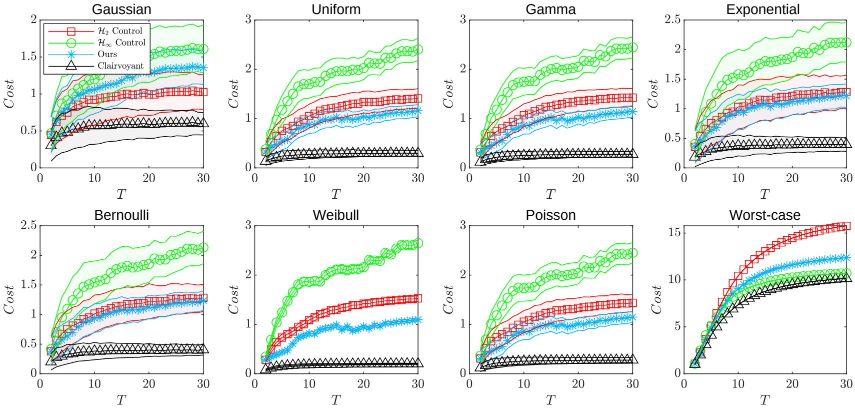
$$A_t = 0.85\begin{bmatrix} 0.7 & 0.2 & 0 \\ 0.3 & 0.7 & -0.1 \\ 0 & -0.2 & 0.8 \end{bmatrix}, B_t = \begin{bmatrix} 1 & 0.2 \\ 2 & 0.3 \\ 1.5 & 0.5 \end{bmatrix}, C_t = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, & t = \{1, 3, \ldots\}; \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & t = \{2, 4, \ldots\}, \end{cases}$$

- $-\mathbf{1}_{3\times 1} \leq w_t \leq \mathbf{1}_{3\times 1}$ and $-\mathbf{1}_{3\times 1} \leq e_t \leq \mathbf{1}_{3\times 1}$ sampled from various distributions;
- safety constraints: $-5 \times \mathbf{1}_{3\times 1} \leq x_t \leq 5 \times \mathbf{1}_{3\times 1}$, and $-5 \times \mathbf{1}_{3\times 1} \leq u_t \leq 5 \times \mathbf{1}_{3\times 1}$;
- quadratic loss function: $c_t(x_{t+1}, u_t) = \|x_{t+1}\|^2 + \|u_t\|^2$;
- total iteration $T = \{2, \ldots, 30\}$;
- comparison with safe H_2 and H_∞.[5]

[5] Anderson et al., ARC '19, Martin et al., L4DC '22

Result:

- Our method lies between H_2 and H_∞ under Gaussian and worst-case noise;
- Our method outperforms H_2 and H_∞ under all other noise;



Setup:

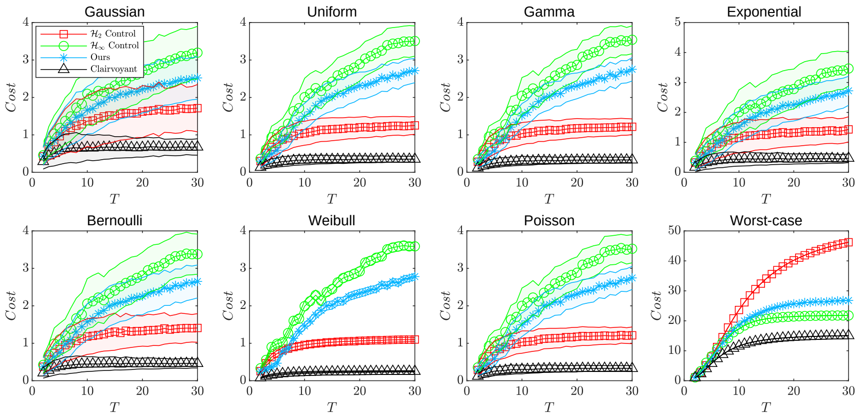
- linear system:

$$A_t = 1.05 \begin{bmatrix} 0.7 & 0.2 & 0 \\ 0.3 & 0.7 & -0.1 \\ 0 & -0.2 & 0.8 \end{bmatrix}, B_t = \begin{bmatrix} 1 & 0.2 \\ 2 & 0.3 \\ 1.5 & 0.5 \end{bmatrix}, C_t = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, & t = \{1, 3, \ldots\}; \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & t = \{2, 4, \ldots\}, \end{cases}$$

- $-\mathbf{1}_{3\times1} \leq w_t \leq \mathbf{1}_{3\times1}$ and $-\mathbf{1}_{3\times1} \leq e_t \leq \mathbf{1}_{3\times1}$ sampled from various distributions;
- safety constraints: $-30 \times \mathbf{1}_{3\times1} \leq x_t \leq 30 \times \mathbf{1}_{3\times1}$, and $-30 \times \mathbf{1}_{3\times1} \leq u_t \leq 30 \times \mathbf{1}_{3\times1}$;
- quadratic loss function: $c_t(x_{t+1}, u_t) = \|x_{t+1}\|^2 + \|u_t\|^2$;
- total iteration $T = \{2, \ldots, 30\}$;

Result:

- Our method lies between H_2 and H_∞ under all tested noise.



Numerical Evaluation on Hovering Quadrotor

Setup:

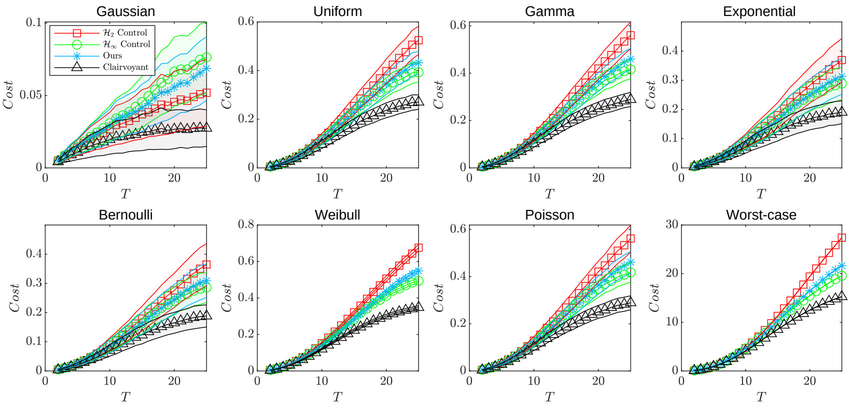
- linearized quadrotor system:

$$A_t = \begin{bmatrix} \mathbf{I}_3 & 0.1 \times \mathbf{I}_3 \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}, B_t = \begin{bmatrix} -\frac{4.91}{100} & 0 & 0 \\ 0 & \frac{4.91}{100} & 0 \\ 0 & 0 & \frac{1}{200} \\ -\frac{98.1}{100} & 0 & 0 \\ 0 & \frac{98.1}{100} & 0 \\ 0 & 0 & \frac{1}{10} \end{bmatrix}., C_t = \begin{cases} \begin{bmatrix} \mathbf{I}_3 & \mathbf{0}_3 \end{bmatrix}, & t = \{1, 4, \ldots\}, \\ \begin{bmatrix} \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix}, & t = \{2, 3, 5, 6 \ldots\}. \end{cases}$$

- $-0.1 \times \mathbf{1}_{3\times1} \leq w_t \leq 0.1 \times \mathbf{1}_{3\times1}$ and $-0.1 \times \mathbf{1}_{3\times1} \leq e_t \leq 0.1 \times \mathbf{1}_{3\times1}$ sampled from various distributions;
- safety constraints: $-5 \times \mathbf{1}_{3\times1} \leq x_t \leq 5 \times \mathbf{1}_{3\times1}$, and $[-\pi \;\; -\pi \;\; -20]^\top \leq u_t \leq [\pi \; \pi \; 20]^\top$;
- quadratic loss function: $c_t(x_{t+1}, u_t) = \|x_{t+1}\|^2 + \|u_t\|^2$;
- total iteration $T = \{2, \ldots, 25\}$;

Result:

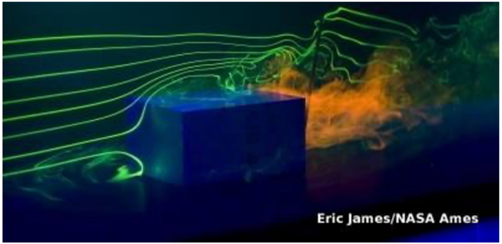
- Our method lies between H_2 and H_∞ under all tested noise.



Summary

Regret optimal control algorithm that

- guarantees safety for partially-observed linear time-varying systems, and
- provides worst-case dynamic regret performance guarantees.



Next steps:

- unknown C_t over horizon T;
- safe non-linear control;[5]
- distributed multi-robot systems.

[5] Zhou, Song, and Tzoumas. Safe Non-Stochastic Control of Control-Affine Systems: An Online Convex Optimization Approach, IEEE Robotics and Automation Letters (RA-L) '23